Maximally superintegrable Smorodinsky-Winternitz systems on the N-dimensional sphere and hyperbolic spaces[1]

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Abstract. The classical Smorodinsky–Winternitz systems on the ND sphere, Euclidean and hyperbolic spaces $\mathbb{S}^N$, $\mathbb{E}^N$ and $\mathbb{H}^N$ are simultaneously approached starting from the Lie algebras $\mathfrak{so}_\kappa(N+1)$, which include a parametric dependence on the curvature. General expressions for the Hamiltonian and its integrals of motion are given in terms of intrinsic geodesic coordinate systems. Each Lie algebra generator gives rise to an integral of motion, so that a set of $N(N+1)/2$ integrals is obtained. Furthermore, $2N-1$ functionally independent ones are identified which, in turn, shows that the well known maximal superintegrability of the Smorodinsky–Winternitz system on $\mathbb{E}^N$ is preserved when curvature arises. On both $\mathbb{S}^N$ and $\mathbb{H}^N$, the resulting system can be interpreted as a superposition of an "actual" oscillator and N "ideal" oscillators (for the sphere, these are alike the actual ones), which can also be understood as N "centrifugal terms"; this is the form seen in the Euclidean limiting case.

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1. Introduction

Superintegrable systems on the two- and three-dimensional (3D) Euclidean spaces have been classified in [6, 25], and also extended to the 2D and 3D spheres [11] as well as to the 2D hyperbolic plane [19, 20] and 3D hyperbolic space [12]. Recent classifications of superintegrable systems for these 2D Riemannian spaces have been presented in [18, 21, 24, 27]. In particular, in the 2D sphere $\mathbb{S}^2$ there are two (maximally) superintegrable families: the harmonic oscillator and the Kepler–Coulomb potential, both of them with some "additional" terms. Let us consider $\mathbb{S}^2$ as embedded through $s_0^2+s_1^2+s_2^2=1$ in an ambient space $\mathbb{R}^3=(s_0,s_1,s_2)$; we set the geodesic polar coordinates (r,θ) such that $s_0=\cos r$, $s_1=\sin r\cos\theta$, $s_2=\sin r\sin\theta$. Following the notation and results given in [27], the first classical superintegrable family on $\mathbb{S}^2$ is given by

$$\begin{aligned}\mathcal{U}_{\rm ho}&=\beta_0\left(\frac{s_1^2+s_2^2}{s_0^2}\right)+\frac{\beta_1}{s_1^2}+\frac{\beta_2}{s_2^2}\\&=\beta_0\tan^2 r+\frac{\beta_1}{\sin^2 r\cos^2\theta}+\frac{\beta_2}{\sin^2 r\sin^2\theta},\end{aligned}\tag{1.1}$$

where β_i are real constants. The first term, $\tan^2 r$, is the *spherical oscillator* or Higgs potential [16, 26]. Under contraction to the Euclidean plane, this reduces to the usual harmonic oscillator, r^2, while the two remaining terms give rise to two "centrifugal barriers". However, very recently, all the three terms in $\mathcal{U}_{\rm ho}$ have been interpreted as spherical oscillators with different centers [29].

On the other hand, the second superintegrable family on $\mathbb{S}^2$ turns out to be

$$\begin{aligned}\mathcal{U}_{\rm KC}&=\beta_0\,\frac{s_0}{\sqrt{s_1^2+s_2^2}}+\beta_1\,\frac{s_1}{s_2^2\sqrt{s_1^2+s_2^2}}+\frac{\beta_2}{s_2^2}\\&=\beta_0\,\frac{1}{\tan r}+\beta_1\,\frac{\cos\theta}{\sin^2 r\sin^2\theta}+\frac{\beta_2}{\sin^2 r\sin^2\theta},\end{aligned}\tag{1.2}$$

where $1/\tan r$ is the "spherical" Kepler–Coulomb potential, first studied by Schrödinger [30]; the two potentials $\tan^2 r$ and $1/\tan r$ are mutually related [22, 23].

The aim of this contribution is to study the maximal superintegrability of the generalization of the potential (1.1) on the ND spaces $\mathbb{S}^N$, $\mathbb{E}^N$ and $\mathbb{H}^N$ from a group theoretical standpoint. This family, depending on the curvature κ as a parameter, includes for $\kappa=0$ the well known maximally superintegrable Euclidean Smorodinsky–Winternitz (SW) system [7–10], the Hamiltonian of which reads

$$\mathcal{H}=\frac{1}{2}\sum_{i=1}^{N}\left(p_i^2+2\beta_0 q_i^2+\frac{2\beta_i}{q_i^2}\right),\tag{1.3}$$

where $\sum_i q_i^2\equiv r^2$ is the harmonic oscillator potential and each $1/q_i^2$ is a "centrifugal

term". Two sets of integrals of motion for $\mathcal{H}$ are given by $(i<j;\ i,j=1,\dots,N)$:

$$
\begin{aligned}
I_{0i} &= \tilde{P}_i^2 + 2\beta_0 q_i^2 + 2\beta_i \frac{1}{q_i^2}, \quad \text{with} \quad \tilde{P}_i = p_i, \\
I_{ij} &= \tilde{J}_{ij}^2 + 2\beta_i \frac{q_j^2}{q_i^2} + 2\beta_j \frac{q_i^2}{q_j^2}, \quad \text{with} \quad \tilde{J}_{ij} = q_i p_j - q_j p_i.
\end{aligned} \tag{1.4}
$$

The first set comes from the separability of the Hamiltonian $2\mathcal{H} = \sum_i I_{0i}$, while the second one has a term quadratic in momenta (the square of the components of the angular momentum tensor) plus some additional terms depending on coordinates. The functions $\tilde{P}_i$, $\tilde{J}_{ij}$ close in phase space the commutation relations of the Euclidean algebra $\mathfrak{iso}(N)$ with respect to the canonical Lie–Poisson bracket:

$$
\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right). \tag{1.5}
$$

The structure of this contribution is as follows. The next section contains the details required (related to their maximal groups of isometries) on the three ND classical Riemannian spaces with constant curvature κ. Geodesic motion is obtained in section 3 starting from the metric on these spaces; this is the kinetic energy term to which possible potentials can be added. The generalization of the SW family to these spaces (any κ) is performed in section 4. General expressions for the Hamiltonian and its integrals of motion are explicitly given both in terms of the (Weierstrass) coordinates in an ambient linear space $\mathbb{R}^{N+1}$ as well as by means of two sets of intrinsic coordinates, the non-zero curvature versions of the Euclidean Cartesian and polar ones. Moreover, $2N-1$ integrals, including the Hamiltonian, are shown to be functionally independent, thus proving that the SW system on curved spaces is also maximally superintegrable [4]. As an example, we apply in the section 5 the general expressions to the $N=4$ case.

2. The sphere $\mathbb{S}^N$, Euclidean $\mathbb{E}^N$, and hyperbolic $\mathbb{H}^N$ spaces

Let $\mathfrak{so}_\kappa(N+1)$ be the Lie algebra of the motion group $SO_\kappa(N+1)$ on a generic ND real Riemannian space with constant curvature κ, denoted $S^N_{[\kappa]}$. In the basis $\{J_{0i}\equiv P_i, J_{ij}\}$ $(i,j=1,\dots,N;\ i<j)$, the non-vanishing commutation relations of $\mathfrak{so}_\kappa(N+1)$ are given by

$$
\begin{aligned}
&[J_{ij},J_{ik}] = J_{jk}, \qquad [J_{ij},J_{jk}] = -J_{ik}, \qquad [J_{ik},J_{jk}] = J_{ij}, \\
&[J_{ij},P_i] = P_j, \qquad [J_{ij},P_j] = -P_i, \qquad [P_i,P_j] = \kappa J_{ij},
\end{aligned} \tag{2.1}
$$

with $i<j<k$. The Lie algebra $\mathfrak{so}_\kappa(N+1)$ is isomorphic to either $\mathfrak{so}(N+1)$ for $\kappa>0$, $\mathfrak{iso}(N)$ for $\kappa=0$, or $\mathfrak{so}(N,1)$ for $\kappa<0$. Notice that by scaling the generators P_i, any

value of κ can always be reduced to either $+1$, 0 or -1. The involutive automorphism defined by

$$\Theta \ : \ J_{ij} \longrightarrow J_{ij}, \qquad P_i \longrightarrow -P_i, \qquad i,j = 1,\ldots,N, \tag{2.2}$$

provides the following Cartan decomposition of $\mathfrak{so}_\kappa(N+1)$:

$$\mathfrak{so}_\kappa(N+1) = \mathfrak{h} \oplus \mathfrak{p}, \qquad \mathfrak{h} = \langle J_{ij}\rangle = \mathfrak{so}(N), \qquad \mathfrak{p} = \langle P_i\rangle. \tag{2.3}$$

The generators invariant under Θ span the subalgebra $\mathfrak{h}$ of the Lie subgroup $H \simeq SO(N)$, so that $S^N_{[\kappa]} = SO_\kappa(N+1)/SO(N)$ is a family, parametrized by κ, of ND symmetric homogeneous rank-one spaces [13]. The generators J_{ij} leave an *origin* point $\mathcal{O}$ invariant, thus acting as rotations around $\mathcal{O}$, while the remaining P_i generate translations that move $\mathcal{O}$ along N basic geodesics l_i orthogonal at $\mathcal{O}$. The space $S^N_{[\kappa]}$ comprises the three classical Riemannian spaces:

$$\begin{array}{lll}
\kappa > 0, & \text{Sphere}, & S^N_{[+]} \equiv SO(N+1)/SO(N) \equiv \mathbb{S}^N; \\
\kappa = 0, & \text{Euclidean}, & S^N_{[0]} \equiv ISO(N)/SO(N) \equiv \mathbb{E}^N; \\
\kappa < 0, & \text{Hyperbolic}, & S^N_{[-]} \equiv SO(N,1)/SO(N) \equiv \mathbb{H}^N.
\end{array}$$

The curvature κ can also be interpreted as a graded contraction parameter coming from the $\mathbb{Z}_2$-grading of $\mathfrak{so}_\kappa(N+1)$ determined by Θ; the value $\kappa = 0$ corresponds to the contraction $\mathbb{S}^N \to \mathbb{E}^N \leftarrow \mathbb{H}^N$ around a point (the origin $\mathcal{O}$) of the spaces.

The Killing–Cartan form of $\mathfrak{so}_\kappa(N+1)$, $g^{KC}(J_{ij}, J_{kl}) = \mathrm{Trace}(\mathrm{ad}\, J_{ij} \cdot \mathrm{ad}\, J_{kl})$, is diagonal with the following non-zero elements

$$g^{KC}(P_i, P_i) = -2(N-1)\kappa, \qquad g^{KC}(J_{ij}, J_{ij}) = -2(N-1); \tag{2.4}$$

hence the restriction of g^{KC} to the subspace $\mathfrak{p}$, can be written as:

$$g^{KC}\big|_{\mathfrak{p}} = -2(N-1)\kappa\, g\big|_{\mathcal{O}}, \qquad g\big|_{\mathcal{O}}(P_i, P_j) = \delta_{ij}, \tag{2.5}$$

where $g\big|_{\mathcal{O}}$ is to be considered as the metric in the tangent space $\mathfrak{p}$ at $\mathcal{O}$, represented by the $N \times N$ matrix $\mathrm{diag}\,(+,+,\ldots,+)$. This metric can be translated to all points of $S^N_{[\kappa]}$ by the group action. Even if g^{KC} vanishes in the Euclidean case, the choice $g \propto g^{KC}/\kappa$ ensures a non-degenerate metric in $S^N_{[\kappa]}$ $\forall\kappa$.

2.1. Vector model and Weierstrass coordinates

The *vector representation* of $\mathfrak{so}_\kappa(N+1)$ is given by $(N+1)\times(N+1)$ real matrices fulfilling (2.1):

$$P_i = -\kappa\, e_{0i} + e_{i0}, \qquad J_{ij} = -e_{ij} + e_{ji}, \tag{2.6}$$

where e_{ij} is the matrix with a single non-zero entry 1, at row i and column j. Their exponential gives rise to the following one-parametric subgroups of $SO_\kappa(N+1)$:

$$
\begin{aligned}
\exp\{xP_i\} &= \sum_{s=1;s\neq i}^{N} e_{ss} + \mathrm{C}_\kappa(x)e_{00} + \mathrm{C}_\kappa(x)e_{ii} - \kappa\,\mathrm{S}_\kappa(x)e_{0i} + \mathrm{S}_\kappa(x)e_{i0} \\
&= \mathbb{I} + P_i\,\mathrm{S}_\kappa(x) + P_i^2\,\mathrm{V}_\kappa(x), \\
\exp\{xJ_{ij}\} &= \sum_{s=0;s\neq i,j}^{N} e_{ss} + \cos x\, e_{ii} + \cos x\, e_{jj} - \sin x\, e_{ij} + \sin x\, e_{ji} \\
&= \mathbb{I} + J_{ij}\sin x + J_{ij}^2(1-\cos x),
\end{aligned}
\tag{2.7}
$$

where $\mathbb{I}$ is the $(N+1)\times(N+1)$ identity matrix. The curvature-dependent cosine $\mathrm{C}_\kappa(x)$ and sine $\mathrm{S}_\kappa(x)$ functions are defined by [3]:

$$
\begin{aligned}
\mathrm{C}_\kappa(x) &= \sum_{l=0}^{\infty}(-\kappa)^l\frac{x^{2l}}{(2l)!} = \begin{cases} \cos\sqrt{\kappa}\,x, & \kappa>0; \\ 1, & \kappa=0; \\ \cosh\sqrt{-\kappa}\,x, & \kappa<0. \end{cases} \\
\mathrm{S}_\kappa(x) &= \sum_{l=0}^{\infty}(-\kappa)^l\frac{x^{2l+1}}{(2l+1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x, & \kappa>0; \\ x, & \kappa=0; \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x, & \kappa<0. \end{cases}
\end{aligned}
\tag{2.8}
$$

From them, we define the "versed sine" or versine $\mathrm{V}_\kappa(x)$ and the tangent $\mathrm{T}_\kappa(x)$:

$$
\mathrm{V}_\kappa(x) = \frac{1}{\kappa}(1-\mathrm{C}_\kappa(x)), \qquad \mathrm{T}_\kappa(x) = \frac{\mathrm{S}_\kappa(x)}{\mathrm{C}_\kappa(x)}. \tag{2.9}
$$

These κ-trigonometric functions coincide with the usual elliptic (circular) and hyperbolic ones for $\kappa=+1$ and $\kappa=-1$, respectively. The "flat" case with $\kappa=0$ gives the parabolic functions: $\mathrm{C}_0(x)=1$, $\mathrm{S}_0(x)=x$ and $\mathrm{V}_0(x)=x^2/2$. In this sense, they can be interpreted as the *"curvature κ-deformed versions"* of 1, x, and x^2. All the known trigonometric relations (necessary in the further computations) also extend for these κ-functions [14, 15] such as, for instance,

$$
\mathrm{C}_\kappa^2(x) + \kappa\,\mathrm{S}_\kappa^2(x) = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{C}_\kappa(x) = -\kappa\,\mathrm{S}_\kappa(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{S}_\kappa(x) = \mathrm{C}_\kappa(x).
$$

In the vector representation (2.6), any generator X satisfies the relation

$$
X^T\mathbb{I}_\kappa + \mathbb{I}_\kappa X = 0, \qquad \mathbb{I}_\kappa = e_{00} + \kappa\sum_{i=1}^{N} e_{ii} = \mathrm{diag}\,(1,\kappa,\ldots,\kappa), \tag{2.10}
$$

(X^T is the transpose matrix of X), so that any element $R \in SO_\kappa(N+1)$ verifies $R^T \mathbb{I}_\kappa R = \mathbb{I}_\kappa$. In this way, $SO_\kappa(N+1)$ can be seen as a group of linear isometries of the bilinear form $\mathbb{I}_\kappa$ acting on $\mathbb{R}^{N+1} = (s_0, s_1, \ldots, s_N)$ via matrix multiplication. The action of $SO_\kappa(N+1)$ on $\mathbb{R}^{N+1}$ is linear but not transitive, since it conserves the quadratic form

$$s_0^2 + \kappa \sum_{i=1}^{N} s_i^2$$

provided by $\mathbb{I}_\kappa$, and $H \simeq SO(N) = \langle J_{ij} \rangle$ (2.3) is the isotopy subgroup of the point $\mathcal{O} = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$ which will be taken as the *origin* in the space $S^N_{[\kappa]}$. The action becomes transitive if we restrict to the orbit in $\mathbb{R}^{N+1}$ of the point $\mathcal{O}$, which is contained in the "sphere" Σ:

$$\Sigma \equiv s_0^2 + \kappa \sum_{i=1}^{N} s_i^2 = 1, \tag{2.11}$$

which reproduces the whole hypersphere, two hyperplanes and a two-sheeted hyperboloid for $\kappa >, =, < 0$, respectively. This orbit is identified with the space $S^N_{[\kappa]}$, and $(s_0, s_1, \ldots, s_N)$, fulfilling the "sphere" constraint (2.11), are called *Weierstrass coordinates.* In terms of these, the metric on $S^N_{[\kappa]}$ comes from the flat ambient metric in $\mathbb{R}^{N+1}$ divided by κ and restricted to Σ:

$$\mathrm{d}\sigma^2 = \frac{1}{\kappa} \left(\mathrm{d}s_0^2 + \kappa \sum_{i=1}^{N} \mathrm{d}s_i^2 \right) \Bigg|_\Sigma = \sum_{i=1}^{N} \mathrm{d}s_i^2 + \kappa \, \frac{\left(\sum_{i=1}^{N} s_i \mathrm{d}s_i \right)^2}{1 - \kappa \sum_{i=1}^{N} \mathrm{d}s_i^2}, \tag{2.12}$$

which reduces to the Euclidean one for $\kappa = 0$. A differential realization of the generators as first-order vector fields in $\mathbb{R}^{N+1}$ with $\partial_i = \partial / \partial s_i$ is (see (2.6)):

$$P_i = \kappa \, s_i \partial_0 - s_0 \partial_i, \qquad J_{ij} = s_j \partial_i - s_i \partial_j. \tag{2.13}$$

2.2. Geodesic coordinate systems

Let us consider in the vector model the origin $\mathcal{O} = (1, 0, \ldots, 0)$ and a generic point $Q \in S^N_{[\kappa]}$ with Weierstrass coordinates $\mathbf{s} = (s_0, s_1, \ldots, s_N) \in \mathbb{R}^{N+1}$. Starting from $\mathcal{O}$, the point Q can be reached in different ways through the action of N one-parametric subgroups (2.7), that is, by means of motions in the space $S^N_{[\kappa]}$. In this way, we introduce coordinates which are intrinsic quantities to the space $S^N_{[\kappa]}$ itself, the associated Weierstrass coordinates automatically fulfilling the condition (2.11). We shall consider two possibilities.

2.2.1. *Geodesic parallel coordinates* $a = (a_1, \ldots, a_N)$.

We move the origin $\mathcal{O}$ by using the N translations subgroups as

$$\mathbf{s}(a) = \exp(a_1 P_1)\, \exp(a_2 P_2) \ldots \exp(a_N P_N)\, \mathcal{O}$$

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \\ s_N \end{pmatrix} = \begin{pmatrix} \mathrm{C}_\kappa(a_1)\, \mathrm{C}_\kappa(a_2)\, \mathrm{C}_\kappa(a_3) \ldots\, \mathrm{C}_\kappa(a_N) \\ \mathrm{S}_\kappa(a_1)\, \mathrm{C}_\kappa(a_2)\, \mathrm{C}_\kappa(a_3) \ldots\, \mathrm{C}_\kappa(a_N) \\ \mathrm{S}_\kappa(a_2)\, \mathrm{C}_\kappa(a_3) \ldots\, \mathrm{C}_\kappa(a_N) \\ \vdots \\ \mathrm{S}_\kappa(a_{N-1})\, \mathrm{C}_\kappa(a_N) \\ \mathrm{S}_\kappa(a_N) \end{pmatrix}. \tag{2.14}$$

Each coordinate a_i is associated to the generator P_i and has dimensions of *length*. If $l_1, l_2, \ldots, l_N$ are the N (oriented) basic geodesics in $S^N_{[\kappa]}$ orthogonal at $\mathcal{O}$, then the first coordinate of a point Q is the geodesic distance a_1 between $\mathcal{O}$ and Q_1 (the orthogonal projection of Q on l_1), measured along l_1. The second coordinate is the distance a_2 between Q_1 and another point Q_2, measured along a geodesic l_2' orthogonal to l_1 through Q_1 and parallel to l_2 (in the sense of parallel transport) and so on, up to reaching Q [15]. This is depicted in figure 1 for $\mathbb{S}^2$ and $\mathbb{H}^2$. By introducing (2.14) in (2.12), we obtain the metric:

$$\mathrm{d}\sigma^2 = \sum_{i=1}^{N-1} \left(\prod_{l=i+1}^{N} \mathrm{C}^2_\kappa(a_l) \right) \mathrm{d}a_i^2 + \mathrm{d}a_N^2. \tag{2.15}$$

From it we may compute the Levi-Civita connection, whose non-zero Christoffel symbols are $(i = 1, \ldots, N-1;\ i+1 \le j \le N)$:

$$\Gamma^i{}_{ij} = -\kappa\, \mathrm{T}_\kappa(a_j), \qquad \Gamma^j{}_{ii} = \kappa\, \mathrm{T}_\kappa(a_j) \prod_{l=i+1}^{j} \mathrm{C}^2_\kappa(a_l). \tag{2.16}$$

2.2.2. *Geodesic polar coordinates* $\theta = (r, \theta_2, \ldots, \theta_N)$.

In this case, we move $\mathcal{O}$ by using the $N-1$ rotations $J_{i\,i+1}$ as well as the first translation $J_{01} \equiv P_1$:

$$\mathbf{s}(\theta) = \exp(\theta_N J_{N-1\,N}) \ldots \exp(\theta_2 J_{12})\, \exp(r P_1)\, \mathcal{O}$$

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \\ s_N \end{pmatrix} = \begin{pmatrix} \mathrm{C}_\kappa(r) \\ \mathrm{S}_\kappa(r) \cos\theta_2 \\ \mathrm{S}_\kappa(r) \sin\theta_2 \cos\theta_3 \\ \vdots \\ \mathrm{S}_\kappa(r) \sin\theta_2 \ldots \sin\theta_{N-1} \cos\theta_N \\ \mathrm{S}_\kappa(r) \sin\theta_2 \ldots \sin\theta_{N-1} \sin\theta_N \end{pmatrix}. \tag{2.17}$$

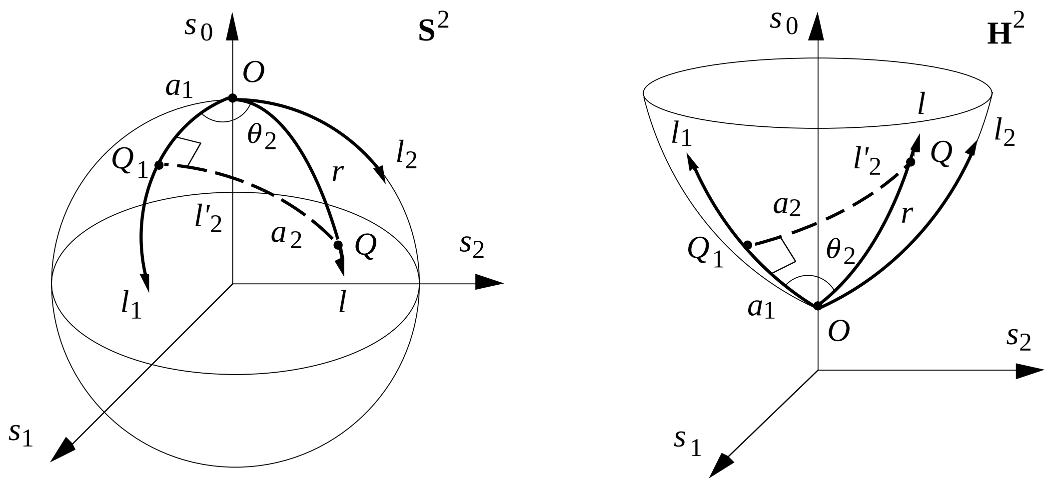

FIGURE 1. Geodesic parallel (a_1, a_2) and polar (r, θ_2) coordinates in $\mathbb{S}^2$ and $\mathbb{H}^2$.

The "radial" coordinate r associated to P_1 has dimensions of *length* and is the distance between $\mathcal{O}$ and Q measured along the geodesic l joining both points. The remaining $\theta_2, \ldots, \theta_N$ are ordinary *angles* parametrizing the orientation of l with respect to the reference flag at $\mathcal{O}$ spanned by $\{l_1, l_2, \ldots, l_{i-1}\}$ (see figure 1 for $N = 2$). On the sphere $\mathbb{S}^N$ of radius R and curvature $\kappa = 1/R^2$, all the usual spherical coordinates are angles, and these differ from the geodesic polar ones only in the first coordinate [28], which is commonly taken as the dimensionless quantity $\theta_1 \equiv r/R$. While these conventional spherical coordinates require an explicit contraction to the Euclidean ones, as done for instance in [17], our choice works for *any* value of κ and when $\kappa = 0$ we recover directly, without any limiting procedure, the polar (and Cartesian) coordinates in $\mathbb{E}^N$.

In polar coordinates the metric (2.12) turns out to be

$$\mathrm{d}\sigma^2 = \mathrm{d}r^2 + \mathrm{S}_\kappa^2(r) \left(\mathrm{d}\theta_2^2 + \sum_{i=3}^{N} \left(\prod_{l=2}^{i-1} \sin^2 \theta_l \right) \mathrm{d}\theta_i^2 \right). \tag{2.18}$$

The components of the Riemann and Ricci tensors can be computed in either geodesic coordinate system; the scalar and the sectional curvatures of $S^N_{[\kappa]}$ are $R = N(N-1)\kappa$ and $K = \kappa$, both constant, respectively. When $\kappa = 0$, the expressions (2.15) and (2.18) give the Euclidean metric

$$\mathrm{d}\sigma^2 = \sum_{i=1}^{N} \mathrm{d}a_i^2 = \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta_2^2 + \sum_{i=3}^{N} \left(\prod_{l=2}^{i-1} \sin^2 \theta_l \right) \mathrm{d}\theta_i^2 \right), \tag{2.19}$$

and all the Christoffel symbols ${\Gamma^i}_{jk}$ (2.16) vanish in parallel coordinates.

3. Geodesic motion on $S^N_{[\kappa]}$ and phase space realization of $\mathfrak{so}_\kappa(N{+}1)$

If we now adopt a dynamical viewpoint, the expressions of the metric in $S^N_{[\kappa]}$ (2.15) and (2.18) provide the kinetic energy $\mathcal{T}$ of a particle in terms of the velocities, generically denoted as $\dot q$ (either $\dot a$ or $\dot\theta$) in the coordinate systems (2.14) and (2.17):

$$2\mathcal{T} = \sum_{i=1}^{N-1}\left(\prod_{l=i+1}^{N}\mathrm{C}^2_\kappa(a_l)\right)\dot a_i^2 + \dot a_N^2 = \dot r^2 + \mathrm{S}^2_\kappa(r)\left(\dot\theta_2^2 + \sum_{i=3}^{N}\left(\prod_{l=2}^{i-1}\sin^2\theta_l\right)\dot\theta_i^2\right), \tag{3.1}$$

which is the Lagrangian $\mathcal{L}\equiv\mathcal{T}$ of the geodesic motion on $S^N_{[\kappa]}$. The canonical momenta, $p=\partial\mathcal{L}/\partial\dot q$, (denoted p,π for $q=a,\theta$) are given by:

$$\begin{aligned} &p_i = \left(\prod_{l=i+1}^{N}\mathrm{C}^2_\kappa(a_l)\right)\dot a_i, \quad i=1,\dots,N-1; \qquad p_N = \dot a_N,\\ &\pi_1 = \dot r, \quad \pi_2 = \mathrm{S}^2_\kappa(r)\dot\theta_2, \quad \pi_j = \mathrm{S}^2_\kappa(r)\left(\prod_{l=2}^{j-1}\sin^2\theta_l\right)\dot\theta_j, \quad j=3,\dots,N. \end{aligned} \tag{3.2}$$

By introducing these momenta in (3.1), we obtain the free Hamiltonian $\mathcal{H}\equiv\mathcal{T}$ in the phase space of motions in $S^N_{[\kappa]}$ expressed in either "parallel" (a,p) or "polar" (θ,π) canonical coordinates and momenta:

$$2\mathcal{T} = \sum_{i=1}^{N-1}\frac{p_i^2}{\prod_{l=i+1}^{N}\mathrm{C}^2_\kappa(a_l)} + p_N^2 = \pi_1^2 + \frac{1}{\mathrm{S}^2_\kappa(r)}\left(\pi_2^2 + \sum_{i=3}^{N}\frac{\pi_i^2}{\prod_{l=2}^{i-1}\sin^2\theta_l}\right). \tag{3.3}$$

On the other hand, the Lie generators are expressed in terms of Weierstrass coordinates as:

$$\tilde P_i(s(q),\dot s(q,p)) = s_0\dot s_i - s_i\dot s_0, \qquad \tilde J_{ij}(s(q),\dot s(q,p)) = s_i\dot s_j - s_j\dot s_i, \tag{3.4}$$

and thus we get an N-particle realization of $\mathfrak{so}_\kappa(N+1)$ in the phase space simply by rewriting everything either in terms of (a,p) or (θ,π). The time derivatives of $\mathbf{s}$ are obtained from either (2.14) in terms of parallel coordinates and velocities $(a,\dot a)$ $(i=1,\dots,N-1)$:

$$\begin{aligned} &\dot s_0 = -\kappa\prod_{m=1}^{N}\mathrm{C}_\kappa(a_m)\sum_{l=1}^{N}\mathrm{T}_\kappa(a_l)\dot a_l,\\ &\dot s_i = \prod_{m=i}^{N}\mathrm{C}_\kappa(a_m)\left(\dot a_i - \kappa\,\mathrm{T}_\kappa(a_i)\sum_{l=i+1}^{N}\mathrm{T}_\kappa(a_l)\dot a_l\right), \qquad \dot s_N = \mathrm{C}_\kappa(a_N)\dot a_N, \end{aligned} \tag{3.5}$$

or from (2.17) in terms of polar coordinates and velocities $(\theta, \dot\theta)$ $(j = 2, \ldots, N-1)$:

$$
\begin{aligned}
&\dot s_0 = -\kappa\,\mathrm{S}_\kappa(r)\dot r, \qquad \dot s_1 = \mathrm{S}_\kappa(r)\sin\theta_2\left(\frac{\dot r}{\mathrm{T}_\kappa(r)\tan\theta_2} - \dot\theta_2\right),\\
&\dot s_j = \mathrm{S}_\kappa(r)\prod_{m=2}^{j+1}\sin\theta_m\left(\frac{\dot r}{\mathrm{T}_\kappa(r)\tan\theta_{j+1}} + \sum_{l=2}^{j}\frac{\dot\theta_l}{\tan\theta_l\tan\theta_{j+1}} - \dot\theta_{j+1}\right),\\
&\dot s_N = \mathrm{S}_\kappa(r)\prod_{m=2}^{N}\sin\theta_m\left(\frac{\dot r}{\mathrm{T}_\kappa(r)} + \sum_{l=2}^{N}\frac{\dot\theta_l}{\tan\theta_l}\right).
\end{aligned}
\tag{3.6}
$$

We now introduce the parametrizations (2.14), (2.17), velocities (3.5), (3.6) as well as the momenta (3.2) in (3.4), obtaining a phase space realization of the generators of $\mathfrak{so}_\kappa(N+1)$ given in "parallel canonical" coordinates by $(i, j = 1, \ldots, N)$:

$$
\begin{aligned}
\tilde P_i &= \prod_{k=1}^{i}\mathrm{C}_\kappa(a_k)\,\mathrm{C}_\kappa(a_i)p_i + \kappa\,\mathrm{S}_\kappa(a_i)\sum_{s=1}^{i}\mathrm{S}_\kappa(a_s)\frac{\prod_{m=1}^{s}\mathrm{C}_\kappa(a_m)}{\prod_{l=s}^{i}\mathrm{C}_\kappa(a_l)}\,p_s,\\
\tilde J_{ij} &= \mathrm{S}_\kappa(a_i)\,\mathrm{C}_\kappa(a_j)\prod_{s=i+1}^{j}\mathrm{C}_\kappa(a_s)p_j - \frac{\mathrm{C}_\kappa(a_i)\,\mathrm{S}_\kappa(a_j)}{\prod_{k=i+1}^{j}\mathrm{C}_\kappa(a_k)}\,p_i\\
&\quad + \kappa\,\mathrm{S}_\kappa(a_i)\,\mathrm{S}_\kappa(a_j)\sum_{s=i+1}^{j}\mathrm{S}_\kappa(a_s)\frac{\prod_{m=i+1}^{s}\mathrm{C}_\kappa(a_m)}{\prod_{l=s}^{j}\mathrm{C}_\kappa(a_l)}\,p_s,
\end{aligned}
\tag{3.7}
$$

and in geodesic polar coordinates and momenta (θ, π) by $(i, j = 1, \ldots, N-1)$:

$$
\begin{aligned}
\tilde P_i &= \frac{\prod_{k=2}^{i+1}\sin\theta_k}{\tan\theta_{i+1}}\,\pi_1 + \sum_{s=2}^{i+1}\frac{\prod_{m=s}^{i+1}\sin\theta_m\,\cos\theta_s\pi_s}{\mathrm{T}_\kappa(r)\tan\theta_{i+1}\prod_{l=2}^{s}\sin\theta_l} - \frac{\pi_{i+1}}{\mathrm{T}_\kappa(r)\prod_{l=2}^{i+1}\sin\theta_l},\\
\tilde P_N &= \prod_{k=2}^{N}\sin\theta_k\,\pi_1 + \sum_{s=2}^{N}\frac{\prod_{m=s}^{N}\sin\theta_m\cos\theta_s}{\mathrm{T}_\kappa(r)\prod_{l=2}^{s}\sin\theta_l}\,\pi_s,\\
\tilde J_{ij} &= \sin\theta_{i+1}\cos\theta_{j+1}\prod_{k=i+1}^{j}\sin\theta_k\,\pi_{i+1} - \frac{\cos\theta_{i+1}\sin\theta_{j+1}}{\prod_{l=i+1}^{j}\sin\theta_l}\,\pi_{j+1}\\
&\quad + \cos\theta_{i+1}\cos\theta_{j+1}\sum_{s=i+1}^{j}\frac{\prod_{m=s}^{j}\sin\theta_m\cos\theta_s}{\prod_{l=i+1}^{s}\sin\theta_l}\,\pi_s,\\
\tilde J_{iN} &= \sin\theta_{i+1}\prod_{k=i+1}^{N}\sin\theta_k\,\pi_{i+1} + \cos\theta_{i+1}\sum_{s=i+1}^{N}\frac{\prod_{m=s}^{N}\sin\theta_m\cos\theta_s}{\prod_{l=i+1}^{s}\sin\theta_l}\,\pi_s.
\end{aligned}
\tag{3.8}
$$

Then the following results follow [4].

Proposition 3.1. *Both sets of generators (3.7) and (3.8) fulfil the commutation rules (2.1) of $\mathfrak{so}_\kappa(N+1)$ with respect to the canonical Poisson bracket (1.5).*

Proposition 3.2. *Any generator (3.7) and (3.8) Poisson-commutes with the kinetic energy $\mathcal{T}$ (3.3).*

The second statement is straightforward as $\mathcal{T}$ is related with a phase space realization of the second-order Casimir $\mathcal{C}$ of $so_\kappa(N+1)$ through

$$2\mathcal{T} = \tilde{\mathcal{C}} = \sum_{i=1}^{N} \tilde{P}_i^2 + \kappa \sum_{i,j=1}^{N} \tilde{J}_{ij}^2. \tag{3.9}$$

In fact, the geodesic motion is maximally superintegrable and its integrals of motion come from any function of a phase space realization of the Lie generators.

4. Smorodinsky–Winternitz system on curved spaces

The problem now is to find which potentials $\mathcal{U}(q)$ can be added to $\mathcal{T}$ in such a manner that the new Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ preserves the maximal superintegrability. This requires to add "some" terms to "some" functions of the generators in order to ensure their involutivity with respect to $\mathcal{H}$.

4.1. Potential

As far as the 2D potential (1.1) is concerned, the generalization to the space $S^N_{[\kappa]}$ is straightforward and reads

$$\mathcal{U}(s(q)) = \beta_0 \frac{\sum_{l=1}^{N} s_l^2}{s_0^2} + \sum_{i=1}^{N} \frac{\beta_i}{s_i^2}, \tag{4.1}$$

which in geodesic parallel and polar coordinates turns out to be

$$\begin{aligned} \mathcal{U} &= \beta_0 \sum_{i=1}^{N} \frac{\mathrm{S}_\kappa^2(a_i)}{\prod_{l=1}^{i} \mathrm{C}_\kappa^2(a_l)} + \sum_{i=1}^{N-1} \frac{\beta_i}{\mathrm{S}_\kappa^2(a_i) \prod_{l=i+1}^{N} \mathrm{C}_\kappa^2(a_l)} + \frac{\beta_N}{\mathrm{S}_\kappa^2(a_N)} \\ &= \beta_0 \, \mathrm{T}_\kappa^2(r) + \frac{1}{\mathrm{S}_\kappa^2(r)} \left(\frac{\beta_1}{\cos^2\theta_2} + \sum_{i=2}^{N-1} \frac{\beta_i}{\cos^2\theta_{i+1} \prod_{l=2}^{i} \sin^2\theta_l} + \frac{\beta_N}{\prod_{l=2}^{N} \sin^2\theta_l} \right). \end{aligned} \tag{4.2}$$

This potential, which coincides with (1.1) for $\kappa = 1$, has been interpreted on $\mathbb{S}^2$ as three Higgs spherical oscillators, whose "centers" are placed at the three vertices of a sphere's octant [29] (recall that the Higgs oscillator has two antipodal "centers"); this interpretation can straightforwardly be extended to the sphere $\mathbb{S}^N$ with $\kappa = 1/R^2 > 0$. Let O_i be the points placed along the basic geodesics l_i and a quadrant apart from $\mathcal{O}$ (each pair taken from $\mathcal{O}, O_i$ are mutually separated a quadrant distance $\pi/(2\sqrt{\kappa}) = R\,\pi/2$ on $\mathbb{S}^N$). If we set $\kappa = 1$ and denote r, r_i the distances between any generic point

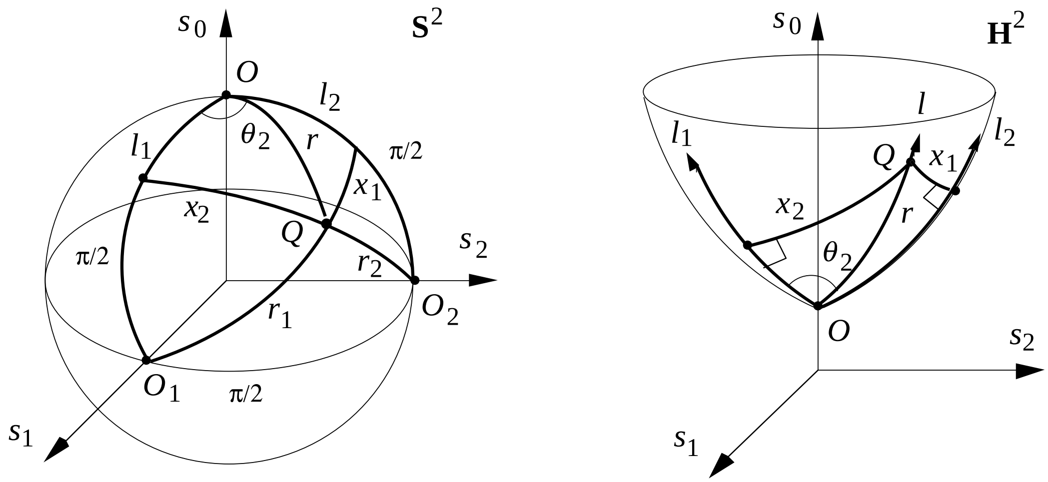

FIGURE 2. Distances involved in the SW potential on $\mathbb{S}^2$ and $\mathbb{H}^2$.

Q and $\mathcal{O}$, O_i along the joining geodesics, we find $s_0 = \cos r$, $s_i = \cos r_i$ $(i = 1, \ldots, N)$, and (4.2) can be rewritten as

$$\mathcal{U} = \beta_0 \tan^2 r + \sum_{i=1}^{N} \frac{\beta_i}{\cos^2 r_i} = \beta_0 \tan^2 r + \sum_{i=1}^{N} \beta_i \tan^2 r_i + \sum_{i=1}^{N} \beta_i, \tag{4.3}$$

which can clearly be recognized under this form as the joint potential due to a set of $N+1$ spherical oscillators whose centers are at the $N+1$ points $\mathcal{O}$, O_i. Alternatively, if $x_i = \pi/2 - r_i$, then each "β_i" term in (4.3) can be described, as usual, as the spherical "centrifugal" barriers $\beta_i / \sin^2 x_i$. Under the contraction $\kappa = 0$, $\mathbb{S}^N \to \mathbb{E}^N$, the Higgs-term gives rise to the "flat" harmonic oscillator $\beta_0 r^2 = \beta_0 \sum_i a_i^2$, while the N remaining oscillators (whose centers would be now "at infinity") leave the Euclidean "centrifugal" barriers $\beta_i / x_i^2 \equiv \beta_i / a_i^2$ as their imprints.

On the hyperbolic space $\mathbb{H}^N$ the potential (4.2) can similarly be interpreted. Let x_i be the distance between the generic point Q and the totally geodesic codimension one submanifold through $\mathcal{O}$ orthogonal to the geodesic l_i. When $\kappa = -1$ we find that $s_0 = \cosh r$, $s_i = \sinh x_i$ $(i = 1, \ldots, N)$, so the potential can be written as:

$$\mathcal{U} = \beta_0 \tanh^2 r + \sum_{i=1}^{N} \frac{\beta_i}{\sinh^2 x_i} = \beta_0 \tanh^2 r + \sum_{i=1}^{N} \frac{\beta_i}{\tanh^2 x_i} - \sum_{i=1}^{N} \beta_i. \tag{4.4}$$

The first term is an "actual" hyperbolic oscillator with center at $\mathcal{O}$ (whose potential is bounded in the whole hyperbolic space), while each of the N remaining terms can be interpreted, as done conventionally, as some kind of hyperbolic "centrifugal" potentials, but also as "ideal" hyperbolic oscillators, whose centers would be beyond infinity, that is, in the exterior region of the hyperbolic space.

4.2. Integrals of motion

By taking into account the results given in [27] for the integrals of motion of (1.1), let us consider the following functions, with quadratic dependence on momenta, given in the ambient space $\mathbb{R}^{N+1}$ by ($i < j,\ i, j = 0, 1, \ldots, N$):

$$I_{ij} = (s_i \dot{s}_j - s_j \dot{s}_i)^2 + 2\beta_i \frac{s_j^2}{s_i^2} + 2\beta_j \frac{s_i^2}{s_j^2}. \tag{4.5}$$

Therefore we have $N(N+1)/2$ phase space functions coming from the Lie generators, which in the geodesic parallel canonical coordinates (3.7) turn out to be

$$\begin{aligned} I_{0i} &= \tilde{P}_i^2 + 2\beta_0 \, \frac{\mathrm{S}_\kappa^2(a_i)}{\prod_{l=1}^{i} \, \mathrm{C}_\kappa^2(a_l)} + 2\beta_i \, \frac{\prod_{l=1}^{i} \, \mathrm{C}_\kappa^2(a_l)}{\mathrm{S}_\kappa^2(a_i)}, \\ I_{ij} &= \tilde{J}_{ij}^2 + 2\beta_i \, \frac{\mathrm{S}_\kappa^2(a_j)}{\mathrm{S}_\kappa^2(a_i) \prod_{l=i+1}^{j} \, \mathrm{C}_\kappa^2(a_l)} + 2\beta_j \, \frac{\mathrm{S}_\kappa^2(a_i) \prod_{l=i+1}^{j} \, \mathrm{C}_\kappa^2(a_l)}{\mathrm{S}_\kappa^2(a_j)}. \end{aligned} \tag{4.6}$$

The same quantities read in the geodesic polar canonical coordinates (3.8):

$$\begin{aligned} I_{0i} &= \tilde{P}_i^2 + 2\beta_0 \, \frac{\mathrm{T}_\kappa^2(r) \prod_{l=2}^{i+1} \sin^2\theta_l}{\tan^2\theta_{i+1}} + 2\beta_i \frac{\tan^2\theta_{i+1}}{\mathrm{T}_\kappa^2(r) \prod_{l=2}^{i+1} \sin^2\theta_l}, \\ I_{0N} &= \tilde{P}_N^2 + 2\beta_0 \ \mathrm{T}_\kappa^2(r) \prod_{l=2}^{N} \sin^2\theta_l + 2\beta_N \frac{1}{\mathrm{T}_\kappa^2(r) \prod_{l=2}^{N} \sin^2\theta_l}, \\ I_{ij} &= \tilde{J}_{ij}^2 + 2\beta_i \, \frac{\cos^2\theta_{j+1} \prod_{l=i+1}^{j} \sin^2\theta_l}{\cos^2\theta_{i+1}} + \frac{2\beta_j \cos^2\theta_{i+1}}{\cos^2\theta_{j+1} \prod_{l=i+1}^{j} \sin^2\theta_l}, \\ I_{iN} &= \tilde{J}_{iN}^2 + 2\beta_i \, \frac{\prod_{l=i+1}^{N} \sin^2\theta_l}{\cos^2\theta_{i+1}} + 2\beta_N \frac{\cos^2\theta_{i+1}}{\prod_{l=i+1}^{N} \sin^2\theta_l}. \end{aligned} \tag{4.7}$$

A first property for these quantities is given by:

Proposition 4.1. *The $N(N+1)/2$ functions given by either (4.6) or (4.7) fulfil $\{I_{ij}, I_{lm}\} = 0$ whenever all four indices $i < j; l < m$, are different.*

From now on we consider the Hamiltonian $\mathcal{H} = \mathcal{T} + \mathcal{U}$ with $\mathcal{T}$ and $\mathcal{U}$ given in (3.3) and (4.2). Then we find that:

Proposition 4.2. *The $N(N+1)/2$ functions either (4.6) or (4.7) are integrals of the motion for the Hamiltonian, that is, $\{I_{ij}, \mathcal{H}\} = 0\ \forall ij$.*

We remark that the property analogous to (3.9) is now given by

$$2\mathcal{H} = \sum_{i=1}^{N} I_{0i} + \kappa \sum_{i,j=1}^{N} I_{ij} + 2\kappa \sum_{i=1}^{N} \beta_i. \tag{4.8}$$

Notice also that when $\kappa = 0$, the Hamiltonian expressed in parallel coordinates and the integrals (4.6) with generators (3.7) directly reduce to the flat SW system characterized by (1.3) and (1.4).

4.3. Maximal superintegrability

The last step is firstly to identify, within the initial set of $N(N+1)/2$ integrals of the motion together with the Hamiltonian, N which are functionally independent and in involution in order to prove the complete integrability of $\mathcal{H}$, and secondly to find out how many are functionally independent thus analyzing its superintegrability.

Let us choose two subsets of $N-1$ integrals $Q^{(l)}$ and $Q_{(l)}$ $(l = 2, \ldots, N)$ obtained starting from the integrals I_{ij} associated to the rotation generators as:

$$Q^{(l)} = \sum_{i,j=1}^{l} I_{ij}, \qquad Q_{(l)} = \sum_{i,j=N-l+1}^{N} I_{ij}, \tag{4.9}$$

which share the element $Q^{(N)} \equiv Q_{(N)}$. The generators that determine the quadratic terms in the momenta in the first set $Q^{(l)}$ are associated to a chain of orthogonal subalgebras within $\mathfrak{h} = \mathfrak{so}(N) = \langle J_{ij} \rangle$ (2.3) starting "upwards" from $\mathfrak{so}(2) = \langle J_{12} \rangle$; likewise for $Q_{(l)}$ but starting "backwards" from $\mathfrak{so}(2) = \langle J_{N-1\,N} \rangle$:

$$\begin{array}{ccc} Q^{(2)} \subset \ldots \subset Q^{(l)} \subset \ldots \subset Q^{(N)} & \qquad & Q_{(N)} \supset \ldots \supset Q_{(l)} \supset \ldots \supset Q_{(2)} \\ \mathfrak{so}(2) \subset \ldots \subset \mathfrak{so}(l) \subset \ldots \subset \mathfrak{so}(N) & \qquad & \mathfrak{so}(N) \supset \ldots \supset \mathfrak{so}(l) \supset \ldots \supset \mathfrak{so}(2) \end{array}$$

We remark that the SW Hamiltonian on $\mathbb{E}^N$, $\mathcal{H}\big|_{\kappa=0}$, can also be constructed through a coalgebra approach [5] by means of N copies of $\mathfrak{sp}(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})$ [1]. In this flat case, each integral of motion $Q^{(l)}\big|_{\kappa=0}$ in Cartesian coordinates *coincides* with an N-particle phase space realization of the k-th order left-coproduct of the Casimir of $\mathfrak{sl}(2,\mathbb{R})$; similarly for $Q_{(l)}\big|_{\kappa=0}$ using the right-coproduct (these are called left- and right-integrals, respectively). Such coalgebra method shows that the N functions $\{Q^{(l)}\big|_{\kappa=0}, \mathcal{H}\big|_{\kappa=0}\}$ are mutually in involution (the right-set as well), and the $2N-2$ functions $\{Q^{(l)}\big|_{\kappa=0}, Q_{(l)}\big|_{\kappa=0}, \mathcal{H}\big|_{\kappa=0}\}$ are functionally independent [2]. For the general case with arbitrary curvature κ, we have the following result.

Theorem 4.3. *(i) The N functions*

$$\{Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)}, \mathcal{H}\}$$

are mutually in involution. The same property holds for the second set

$$\{Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, \mathcal{H}\}.$$

(ii) The $2N-1$ functions

$$\{Q^{(2)}, Q^{(3)}, \ldots, Q^{(N)} \equiv Q_{(N)}, \ldots, Q_{(3)}, Q_{(2)}, I_{0i}, \mathcal{H}\},$$

where i is fixed, are functionally independent, thus $\mathcal{H}$ is a maximally superintegrable Hamiltonian.

Notice that if $\kappa \neq 0$, each $Q^{(l)}$ (or $Q_{(l)}$) can be seen as a smooth function on the curvature:

$$Q^{(l)} = Q^{(l)}\big|_{\kappa=0} + \kappa\, Q_1^{(l)} + o(\kappa^2).$$

The same holds if one introduces a quantum deformation of the Euclidean SW system in terms of a deformation parameter z [1, 2]: superintegrability is preserved and the deformed integrals $Q_z^{(l)}$ (or $Q^z_{(l)}$) can be written as power series of z:

$$Q_z^{(l)} = Q_z^{(l)}\big|_{z=0} + z\, Q_{z,1}^{(l)} + o(z^2).$$

Then, $Q^{(l)}\big|_{\kappa=0} \equiv Q_z^{(l)}\big|_{z=0}$ are the integrals of the SW system on $\mathbb{E}^N$. This suggests some kind of duality between quantum deformations (z) and curvature (κ) [3].

5. The $\mathbf{N = 4}$ case

In what follows we illustrate the general expressions obtained in the previous sections by applying them to the case $N = 4$ in terms of geodesic parallel coordinates and momenta. The SW Hamiltonian on $\mathbb{S}^4$, $\mathbb{E}^4$ and $\mathbb{H}^4$ reads

$$\begin{aligned}
\mathcal{H} = {} & \frac{1}{2}\left(\frac{p_1^2}{\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)\,\mathrm{C}_\kappa^2(a_4)} + \frac{p_2^2}{\mathrm{C}_\kappa^2(a_3)\,\mathrm{C}_\kappa^2(a_4)} + \frac{p_3^2}{\mathrm{C}_\kappa^2(a_4)} + p_4^2 \right) \\
& + \beta_0 \left(\mathrm{T}_\kappa^2(a_1) + \frac{\mathrm{T}_\kappa^2(a_2)}{\mathrm{C}_\kappa^2(a_1)} + \frac{\mathrm{T}_\kappa^2(a_3)}{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)} + \frac{\mathrm{T}_\kappa^2(a_4)}{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)} \right) \\
& + \frac{\beta_1}{\mathrm{S}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)\,\mathrm{C}_\kappa^2(a_4)} + \frac{\beta_2}{\mathrm{S}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)\,\mathrm{C}_\kappa^2(a_4)} \\
& + \frac{\beta_3}{\mathrm{S}_\kappa^2(a_3)\,\mathrm{C}_\kappa^2(a_4)} + \frac{\beta_4}{\mathrm{S}_\kappa^2(a_4)}.
\end{aligned} \tag{5.1}$$

The phase space realization (3.7) for $\mathfrak{so}_\kappa(5)$ is given by four translations

$$
\begin{aligned}
&\tilde P_1 = p_1, \qquad \tilde P_2 = \mathrm{C}_\kappa(a_1) p_2 + \kappa\,\mathrm{S}_\kappa(a_1)\,\mathrm{T}_\kappa(a_2) p_1,\\
&\tilde P_3 = \mathrm{C}_\kappa(a_1)\,\mathrm{C}_\kappa(a_2) p_3 + \kappa\,\mathrm{T}_\kappa(a_3)\left(\frac{\mathrm{S}_\kappa(a_1)}{\mathrm{C}_\kappa(a_2)}\,p_1 + \mathrm{C}_\kappa(a_1)\,\mathrm{S}_\kappa(a_2) p_2\right),\\
&\tilde P_4 = \mathrm{C}_\kappa(a_1)\,\mathrm{C}_\kappa(a_2)\,\mathrm{C}_\kappa(a_3) p_4 + \kappa\,\mathrm{T}_\kappa(a_4)\left(\frac{\mathrm{S}_\kappa(a_1)}{\mathrm{C}_\kappa(a_2)\,\mathrm{C}_\kappa(a_3)}\,p_1\right.\\
&\qquad\qquad \left. + \frac{\mathrm{C}_\kappa(a_1)\,\mathrm{S}_\kappa(a_2)}{\mathrm{C}_\kappa(a_3)}\,p_2 + \mathrm{C}_\kappa(a_1)\,\mathrm{C}_\kappa(a_2)\,\mathrm{S}_\kappa(a_3) p_3\right),
\end{aligned} \tag{5.2}
$$

together with six rotation generators:

$$
\begin{aligned}
&\tilde J_{12} = \mathrm{S}_\kappa(a_1) p_2 - \mathrm{C}_\kappa(a_1)\,\mathrm{T}_\kappa(a_2) p_1,\\
&\tilde J_{13} = \mathrm{S}_\kappa(a_1)\,\mathrm{C}_\kappa(a_2) p_3 - \frac{\mathrm{C}_\kappa(a_1)}{\mathrm{C}_\kappa(a_2)}\,\mathrm{T}_\kappa(a_3) p_1 + \kappa\,\mathrm{S}_\kappa(a_1)\,\mathrm{S}_\kappa(a_2)\,\mathrm{T}_\kappa(a_3) p_2,\\
&\tilde J_{14} = \mathrm{S}_\kappa(a_1)\,\mathrm{C}_\kappa(a_2)\,\mathrm{C}_\kappa(a_3) p_4 - \frac{\mathrm{C}_\kappa(a_1)}{\mathrm{C}_\kappa(a_2)\,\mathrm{C}_\kappa(a_3)}\,\mathrm{T}_\kappa(a_4) p_1\\
&\qquad\qquad + \kappa\,\mathrm{S}_\kappa(a_1)\,\mathrm{T}_\kappa(a_4)\left(\frac{\mathrm{S}_\kappa(a_2)}{\mathrm{C}_\kappa(a_3)} p_2 + \mathrm{C}_\kappa(a_2)\,\mathrm{S}_\kappa(a_3) p_3\right),\\
&\tilde J_{23} = \mathrm{S}_\kappa(a_2) p_3 - \mathrm{C}_\kappa(a_2)\,\mathrm{T}_\kappa(a_3) p_2,\\
&\tilde J_{24} = \mathrm{S}_\kappa(a_2)\,\mathrm{C}_\kappa(a_3) p_4 - \frac{\mathrm{C}_\kappa(a_2)}{\mathrm{C}_\kappa(a_3)}\,\mathrm{T}_\kappa(a_4) p_2 + \kappa\,\mathrm{S}_\kappa(a_2)\,\mathrm{S}_\kappa(a_3)\,\mathrm{T}_\kappa(a_4) p_3,\\
&\tilde J_{34} = \mathrm{S}_\kappa(a_3) p_4 - \mathrm{C}_\kappa(a_3)\,\mathrm{T}_\kappa(a_4) p_3.
\end{aligned} \tag{5.3}
$$

From these generators we construct ten integrals of motion (4.6):

$$
\begin{array}{cccc}
\tilde P_1 & \tilde P_2 & \tilde P_3 & \tilde P_4 \\ \hline
 & \tilde J_{12} & \tilde J_{13} & \tilde J_{14} \\
 & & \tilde J_{23} & \tilde J_{24} \\
 & & & \tilde J_{34}
\end{array}
\quad \Longrightarrow \quad
\begin{array}{cccc}
I_{01} & I_{02} & I_{03} & I_{04} \\ \hline
 & I_{12} & I_{13} & I_{14} \\
 & & I_{23} & I_{24} \\
 & & & I_{34}
\end{array}
$$

Hence from (5.2) we obtain four "translation-like" integrals given by

$$
\begin{aligned}
&I_{01} = \tilde P_1^2 + 2\beta_0\,\mathrm{T}_\kappa^2(a_1) + 2\beta_1\frac{1}{\mathrm{T}_\kappa^2(a_1)},\\
&I_{02} = \tilde P_2^2 + 2\beta_0\frac{\mathrm{T}_\kappa^2(a_2)}{\mathrm{C}_\kappa^2(a_1)} + 2\beta_2\frac{\mathrm{C}_\kappa^2(a_1)}{\mathrm{T}_\kappa^2(a_2)},\\
&I_{03} = \tilde P_3^2 + 2\beta_0\frac{\mathrm{T}_\kappa^2(a_3)}{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)} + 2\beta_3\frac{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)}{\mathrm{T}_\kappa^2(a_3)},\\
&I_{04} = \tilde P_4^2 + 2\beta_0\frac{\mathrm{T}_\kappa^2(a_4)}{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)} + 2\beta_4\frac{\mathrm{C}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)}{\mathrm{T}_\kappa^2(a_4)},
\end{aligned} \tag{5.4}
$$

together with six "rotation-like" ones coming from (5.3):

$$
\begin{aligned}
I_{12} &= \tilde{J}_{12}^2 + 2\beta_1 \frac{\mathrm{T}_\kappa^2(a_2)}{\mathrm{S}_\kappa^2(a_1)} + 2\beta_2 \frac{\mathrm{S}_\kappa^2(a_1)}{\mathrm{T}_\kappa^2(a_2)},\\
I_{13} &= \tilde{J}_{13}^2 + 2\beta_1 \frac{\mathrm{T}_\kappa^2(a_3)}{\mathrm{S}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)} + 2\beta_3 \frac{\mathrm{S}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)}{\mathrm{T}_\kappa^2(a_3)},\\
I_{14} &= \tilde{J}_{14}^2 + 2\beta_1 \frac{\mathrm{T}_\kappa^2(a_4)}{\mathrm{S}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)} + 2\beta_4 \frac{\mathrm{S}_\kappa^2(a_1)\,\mathrm{C}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)}{\mathrm{T}_\kappa^2(a_4)},\\
I_{23} &= \tilde{J}_{23}^2 + 2\beta_2 \frac{\mathrm{T}_\kappa^2(a_3)}{\mathrm{S}_\kappa^2(a_2)} + 2\beta_3 \frac{\mathrm{S}_\kappa^2(a_2)}{\mathrm{T}_\kappa^2(a_3)},\\
I_{24} &= \tilde{J}_{24}^2 + 2\beta_2 \frac{\mathrm{T}_\kappa^2(a_4)}{\mathrm{S}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)} + 2\beta_4 \frac{\mathrm{S}_\kappa^2(a_2)\,\mathrm{C}_\kappa^2(a_3)}{\mathrm{T}_\kappa^2(a_4)},\\
I_{34} &= \tilde{J}_{34}^2 + 2\beta_3 \frac{\mathrm{T}_\kappa^2(a_4)}{\mathrm{S}_\kappa^2(a_3)} + 2\beta_4 \frac{\mathrm{S}_\kappa^2(a_3)}{\mathrm{T}_\kappa^2(a_4)}.
\end{aligned}
\tag{5.5}
$$

Thus, in this case, within the subset of "rotation-like" integrals, we find three "upwards-integrals" $Q^{(2)}, Q^{(3)}, Q^{(4)}$, associated to $\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \mathfrak{so}(4)$, that determine a completely integrable Hamiltonian:

$$
Q^{(2)} = I_{12} \qquad Q^{(3)} = \begin{aligned}[t] I_{12} &+ I_{13} \\ &+ I_{23} \end{aligned} \qquad Q^{(4)} = Q_{(4)} = \begin{aligned}[t] I_{12} + I_{13} &+ I_{14} \\ + I_{23} &+ I_{24} \\ &+ I_{34} \end{aligned}
$$

These three functions together with the two "backwards-integrals" $Q_{(2)}, Q_{(3)}$, associated to $\mathfrak{so}(2) \subset \mathfrak{so}(3)$, and one additional "translation-like" one, say I_{01}, characterize a maximally superintegrable Hamiltonian (5.1):

$$
Q_{(2)} = I_{34} \qquad Q_{(3)} = \begin{aligned}[t] I_{23} &+ I_{24} \\ &+ I_{34} \end{aligned} \qquad I_{01}
$$

To end with, we would like to point out that a similar algebraic construction may likely be applied to the curved version of the "Kepler–Coulomb" superintegrable family (1.2). Furthermore, the consideration of a second contraction parameter, say κ_2, that enables to take into account indefinite metrics of Lorentzian signature [14, 15], would allow one to obtain superintegrable systems on different spacetimes. The application of these maximally superintegrable systems in quantum mechanics also deserves a further study.

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